

# Progression of Calculations

(From concrete to pictorial to abstract methods)

A Parental Guide

CALDER HOUSE SCHOOL

March 2018

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## **Introduction:**

This guidance contains the key methods of calculation that are taught at Calder House School.

The overall aim is that the children we teach should:

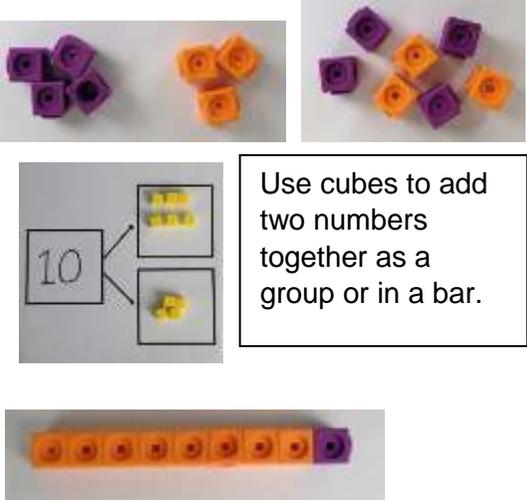
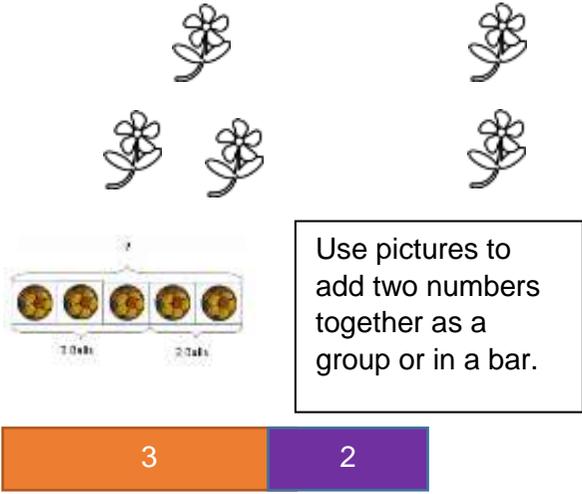
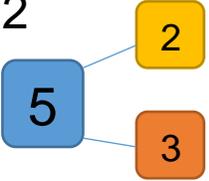
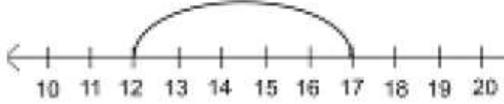
- have a secure knowledge of number facts and a good understanding of the four operations
- make use of diagrams and informal notes to help record steps and part answers so they have a visual reminder of the question
- have an efficient, reliable and formal written method of calculation for each operation that they can apply with confidence when undertaking calculations.

It is important to realise that the children at Calder House School have complex learning profiles, and join us at different times in their education; for this reason, the children may be using a variety of concrete, pictorial and formal written methods. Teachers start from the point of the child's mathematical understanding and use what they already know.

(Please note the terms units and ones are used simultaneously in our teaching at CHS and both can be found in this document).

# Progression in Calculations

## Addition

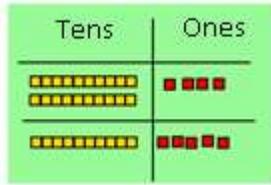
Objective and Strategies	Concrete	Pictorial	Abstract
<p>Combining two parts to make a whole</p>	 <p>Use cubes to add two numbers together as a group or in a bar.</p>	 <p>Use pictures to add two numbers together as a group or in a bar.</p>	<p><math>2 + 3 = 5</math></p> <p><math>5 = 3 + 2</math></p>  <p>Use the part-part whole diagram as shown above to move into the formal written method.</p>
<p>Starting at the bigger number and counting on</p>	 <p>Start with the larger number on the bead string and then count on to the smaller number 1 by 1 to find the answer. <math>12 + 5 = 17</math></p>	<p><math>12 + 5 = 17</math></p>  <p>Start at the larger number on the number line and count on in ones or in one jump to find the answer.</p>	<p><math>5 + 12 = 17</math></p> <p>Place the larger number in your head and count on the smaller number to find your answer.</p>



## Column method- no regrouping

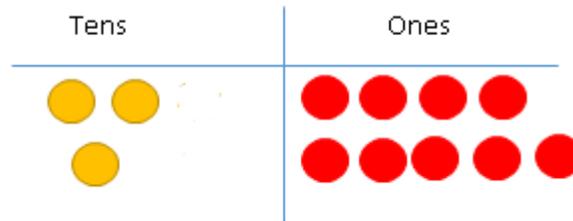
$$24 + 15 =$$

Add together the ones first then add the tens. Use the Base 10 blocks first before moving onto place value counters.



$$30 + 9 = 39$$

After practically using the base 10 blocks and place value counters, children can **DRAW** the counters to help them to solve additions.  $24 + 15 = 39$



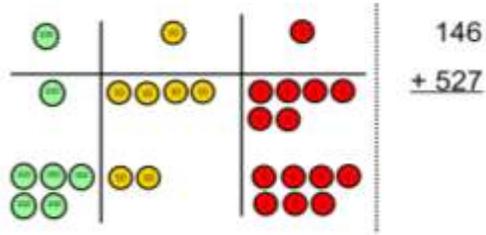
## Calculations

What is  $24 + 15$ ?

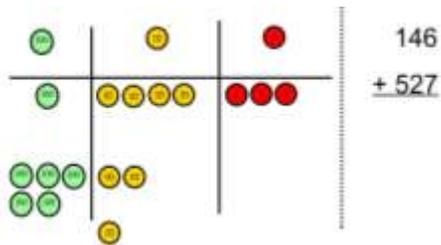
$$\begin{array}{r} 24 \\ + 15 \\ \hline 39 \end{array}$$

## Column method- regrouping

Make both numbers on a place value grid using concrete apparatus



Add up the units and exchange 10 ones for one 10.



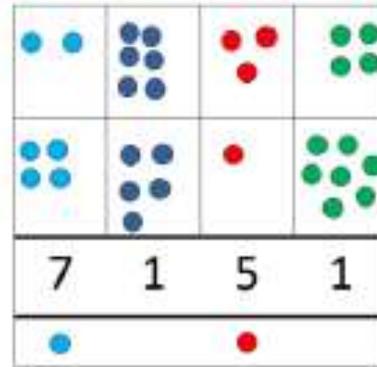
Add up the rest of the columns, exchanging the 10 counters from one column for the next place value column until every column has been added.

This can also be done with Base 10 to help children clearly see that 10 ones equal 1 ten and 10 tens equal 100.

As children move on to decimals, money and decimal place value counters can be used to support learning.

Children can draw a pictorial representation of the columns and place value counters to further support their learning and understanding.

Th H T U



Start by partitioning the numbers before moving on to clearly show the exchange below the addition.

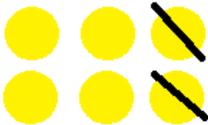
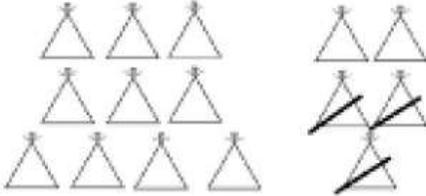
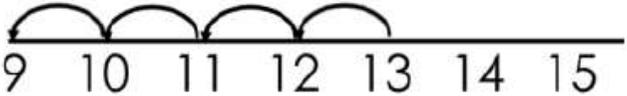
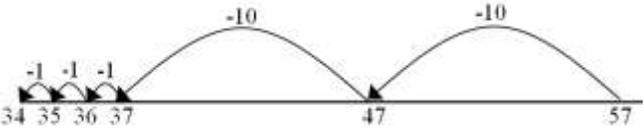
$$\begin{array}{r} 20 + 5 \\ 40 + 8 \\ 60 + 13 = 73 \end{array}$$

As the children move on, introduce decimals. Money can be used here.

$$\begin{array}{r} 536 \\ + 85 \\ \hline 621 \\ 11 \end{array}$$

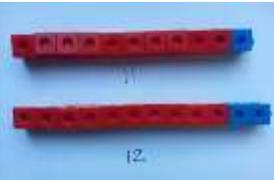
$$\begin{array}{r} \pounds 23.59 \\ \pounds + 7.55 \\ \hline \pounds 31.14 \\ 111 \end{array}$$

## Subtraction

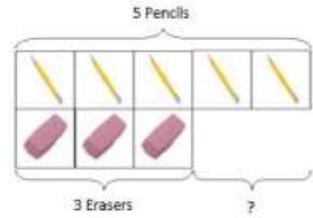
Objective and Strategies	Concrete	Pictorial	Abstract
<p><b>Taking away ones</b></p>	<p>Use physical objects, counters, cubes etc to show how objects can be taken away.</p> <p style="text-align: right;"><math>6 - 2 = 4</math></p> 	<p>Cross out drawn objects to show what has been taken away.</p>  <p style="text-align: center;"><math>15 - 3 = 12</math></p>	<p><math>15 - 3 = 12</math></p> <p><math>6 - 2 = 4</math></p>
<p><b>Counting back</b></p>	<p>Make the larger number in your subtraction. Move the beads along your bead string as you count backwards in ones.</p>  <p><math>13 - 4</math></p> <p>Use counters and move them away from the group as you take them away counting backwards as you go.</p> 	<p>Count back on a number line or number track <math>13 - 4 = 9</math></p>  <p><math>9 \quad 10 \quad 11 \quad 12 \quad 13 \quad 14 \quad 15</math></p> <p>Start at the bigger number and count back towards the smaller number showing the jumps on the number line.</p> <p><math>57 - 23 = 34</math></p>  <p><math>34 \quad 35 \quad 36 \quad 37 \quad 47 \quad 57</math></p>	<p>Put 13 in your head, count back 4. What number are you at? Use your fingers to help.</p> <p><math>13 - 4 = 9</math></p>

### Find the difference

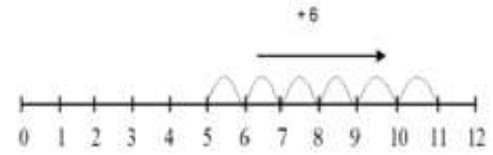
Compare amounts and objects to find the difference.



Use cubes to build towers or make bars to find the difference between 12 and 11.



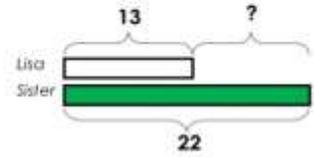
Find the difference between 5 and 3.



Count on to find the difference between 5 and 11.

#### Comparison Bar Models

Lisa is 13 years old. Her sister is 22 years old. Find the difference in age between them.

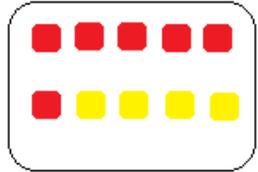


Draw bars to find the difference between 2 numbers.

Hannah has 23 sandwiches, Helen has 15 sandwiches. Find the difference between the numbers of sandwiches.

$$23 - 15 = 8$$

### Parts of the Whole Model



Link to addition- use the part whole model to help explain the inverse between addition and subtraction. If 10 is the whole and 6 is one of the parts. What is the other part?

$$10 - 6 = 4$$

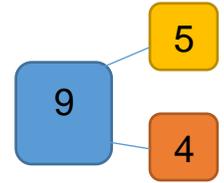
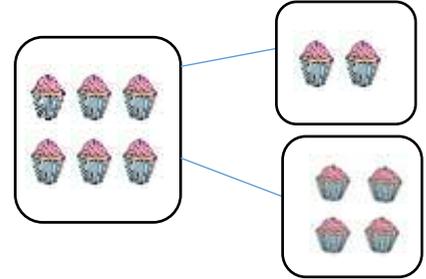
Use a pictorial representation of objects to show parts of the whole model.

$$2 + 4 = 6$$

$$4 + 2 = 6$$

$$6 - 4 = 2$$

$$6 - 2 = 4$$



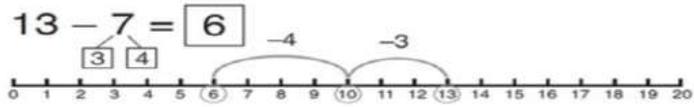
If you know  $4 + 5 = 9$ . What other number sentences can you write?

### Make 10

$$14 - 5 = 9$$



Make 14 on the ten frame. Take away the four first to make 10 and then take away one more so you have taken away 5. You are left with the answer of 9.



Start at 13. Take away 3 to reach 10. Then take away the remaining 4 so you have taken away 7 altogether. You have reached your answer.

$$16 - 8 =$$

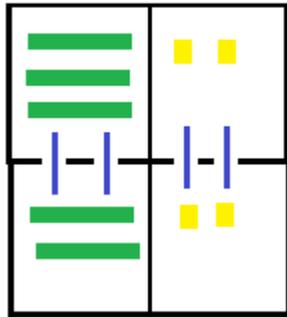
How many do we take off to reach the next 10? (6)

How many do we have left to take off? (2)

$$16 - 8 = 8$$

Column method without regrouping

Tens Ones



Use Base 10 to make the bigger number then take the smaller number away.  
 $54 - 22 = 32$

Draw the Base 10 or place value counters alongside the written calculation to help to show working.

Tens	Ones / Units	Calculations
		$54$
		$- 22$
		$\underline{32}$

			Calculations
			$176 - 64 =$
			$176$
			$- 64$
			$\underline{112}$

This will lead to a clear written column subtraction.

$$\begin{array}{r} 54 \\ - 22 \\ \hline 32 \end{array}$$

# Column method with regrouping

Use place value counters. Start with one exchange before moving onto subtractions with 2 exchanges.

Make the larger number with the place value counters

Calculations
234
- 88
-----
146

Start with the ones, can I take away 8 from 4 easily? I need to exchange one of my tens for ten ones.

Calculations
234
- 88
-----
146

Now I can subtract my ones.

Now look at the tens, can I take away 8 tens easily? I need to exchange one hundred for ten tens.

Calculations
234
- 88
-----
146

Hundreds	Tens	Ones
6	2	6
- 2	- 7	- 5
-----	-----	-----
3	5	1

Draw the counters onto a place value grid and show what you have taken away by crossing the counters out as well as clearly showing the exchanges you make.  
 $626 - 275 = 351$

$$\begin{array}{r} 836 - 254 = 582 \\ \begin{array}{r} \text{h} \quad \text{t} \quad \text{u} \\ 836 \\ - 254 \\ \hline 582 \end{array} \end{array}$$

Children can start their formal written method by partitioning the number into clear place value columns.

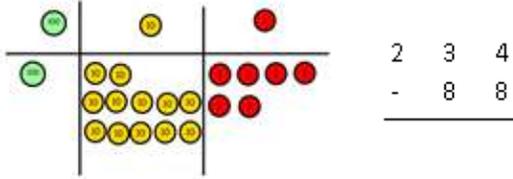
$$\begin{array}{r} 728 - 582 = 146 \\ \begin{array}{r} \text{h} \quad \text{t} \quad \text{u} \\ 728 \\ - 582 \\ \hline 146 \end{array} \end{array}$$

Moving forward the children use a more compact method.

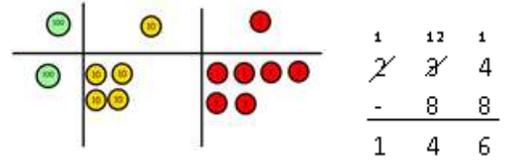
This will lead to an understanding of subtracting any number including decimals.

$$\begin{array}{r} 5 \quad 12 \quad 1 \\ 263.0 \\ - 26.5 \\ \hline 236.5 \end{array}$$

Now I can take away eight tens and complete my subtraction.



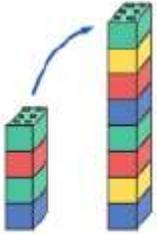
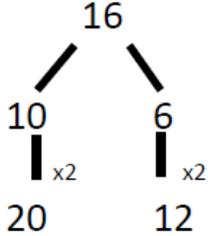
$$\begin{array}{r} 2 \ 3 \ 4 \\ - \ 8 \ 8 \\ \hline \end{array}$$



$$\begin{array}{r} 1 \ 12 \ 4 \\ \cancel{2} \ \cancel{3} \ 4 \\ - \ 8 \ 8 \\ \hline 1 \ 4 \ 6 \end{array}$$

Show children how the concrete method links to the written method alongside your working. Cross out the numbers when exchanging and show where we write our new amount.

## Multiplication

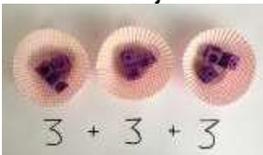
Objective and Strategies	Concrete	Pictorial	Abstract
<b>Doubling</b>	<p>Use practical activities to show how to double a number.</p>  <p>double 4 is 8 <math>4 \times 2 = 8</math></p>	<p>Draw pictures to show how to double a number.</p> <p>Double 4 is 8</p> 	<p>What is double 16?</p>  <p><math>20 + 12 = 32</math></p> <p>Partition a number and then double each part before recombining it back together.</p>

## Counting in multiples (repeated addition)

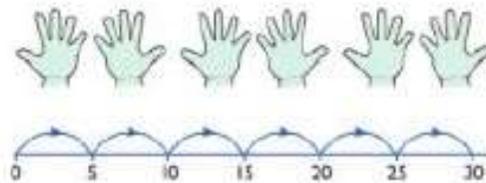


$$4 + 4 + 4 + 4 = 16$$

Count in multiples supported by concrete objects in equal groups.

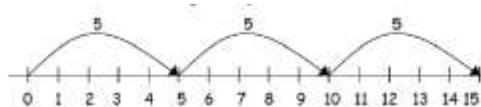


Use different objects to add equal groups.



Use a number line or pictures to continue support in counting in multiples.

There are 3 plates. Each plate has 2 star biscuits on. How many biscuits are there?



$$5 + 5 + 5 = 15$$

Count in multiples of a number aloud.

Write sequences with multiples of numbers.

2, 4, 6, 8, 10

5, 10, 15, 20, 25, 30

Write addition sentences to describe objects and pictures. If I have five pairs of socks. How many socks do I have?

$$2 + 2 + 2 + 2 + 2 = 10$$

$$5 \times 2 = 10$$

Arrays-  
showing  
commutative  
( $3 \times 4 = 4 \times 3$ )  
multiplication

Create arrays using counters/ cubes to show multiplication sentences.

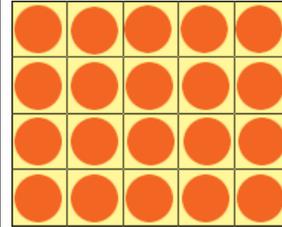
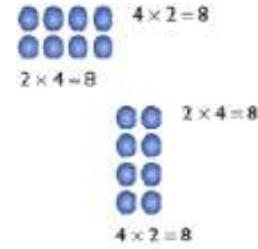
$$4 \times 6 = 6 \times 4$$



$$5 \times 3 = 3 \times 5$$



Draw arrays in different rotations to find **commutative** multiplication sentences.



Link arrays to area of rectangles.

$$5 \times 4 = 4 \times 5$$

Use an array to write multiplication sentences and reinforce repeated addition.



$$5 + 5 + 5 = 15$$

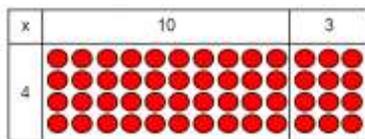
$$3 + 3 + 3 + 3 + 3 = 15$$

$$5 \times 3 = 15$$

$$3 \times 5 = 15$$

## Grid Method

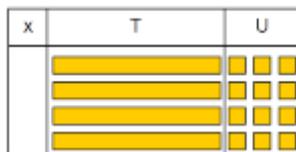
Introduce the link between arrays and the grid method.



4 rows of 10  
4 rows of 3

Move on to using Base 10 to move towards a more compact method.

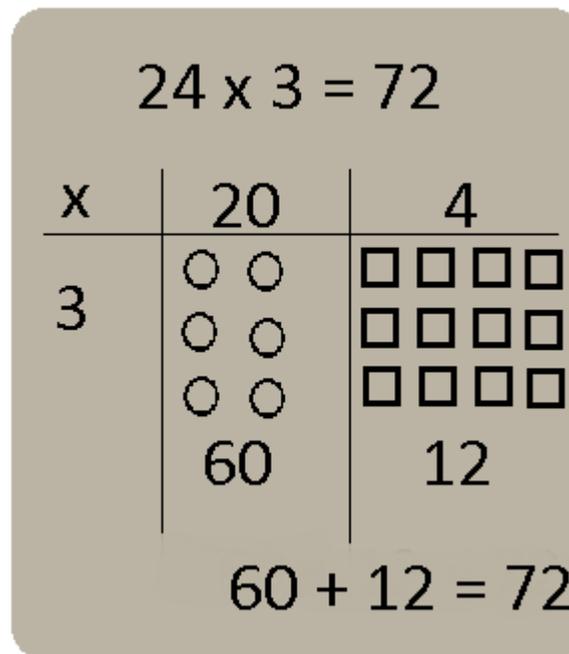
4 rows of 13



Move on to place value counters to show how we are finding groups of a number. We are multiplying by 4 so we need 4 rows.  $4 \times 13 = 52$

Children can represent the work they have done with place value counters in a way that they understand.

They can draw the counters, using colours to show different amounts or just use circles in the different columns to show their thinking as shown below.



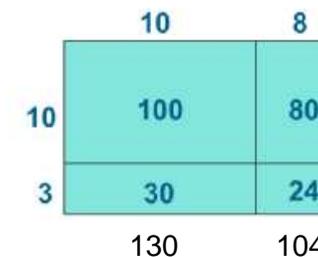
Start with multiplying by one digit numbers and showing the clear addition alongside the grid.

x	30	5
7	210	35

$$210 + 35 = 245$$

Moving forward, multiply by a 2 digit number showing the different rows within the grid method.

Find  $13 \times 18$ .

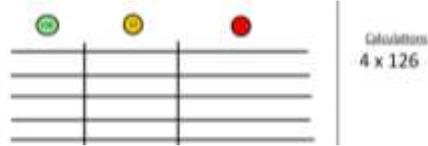


$$13 \times 18 = 130 + 104 = 234$$

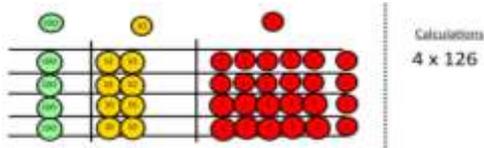
x	1000	300	40	2
18	18000	5400	720	36

$$\begin{aligned} &18000 \ 5400 \ 720 \ 36 \\ = &18 \times 1342 \\ = &18000 + 5400 + 720 + 36 \\ = &24156 \end{aligned}$$

Multiplying a 3-digit number by a 1-digit number

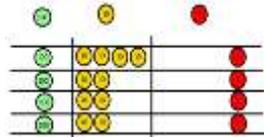


Fill each row with 126.



$$400 + 80 + 24 =$$

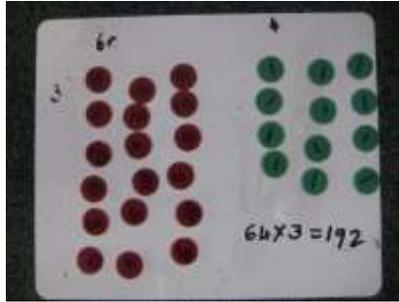
Add up each column, starting with the ones making any exchanges needed.



Then you have your answer = 504

## Column multiplication

Children can continue to be supported by place value counters at the stage of multiplication.



It is important at this stage that they always multiply the ones first and note down their answer followed by the tens which they note below.

$$180 + 12 = 192$$

Bar modelling and number lines can support learners when solving problems with multiplication alongside the formal written methods.

$$\boxed{59} \boxed{59} \boxed{59} \boxed{59} \boxed{59} \boxed{59} \boxed{59} \boxed{59}$$

$$8 \times 59 = 8 \times 60 - 8$$

$$8 \times 6 = 48 \text{ therefore } 8 \times 60 = 480$$

$$480 - 8 = 472$$

$$\text{so } 8 \times 59 = 472$$

Start with long multiplication, reminding the children about lining up their numbers clearly in columns.

If it helps, children can write out what they are solving next to their answer.

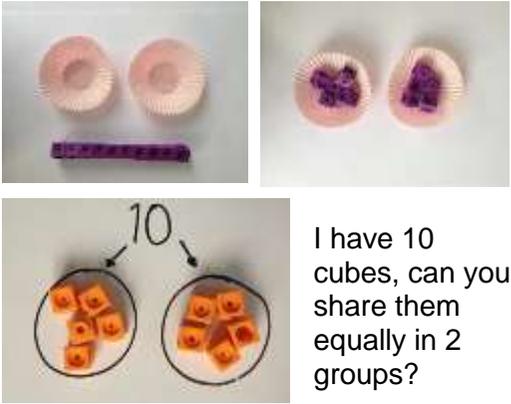
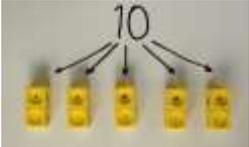
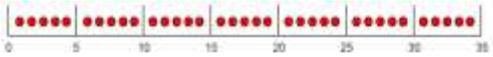
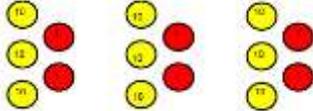
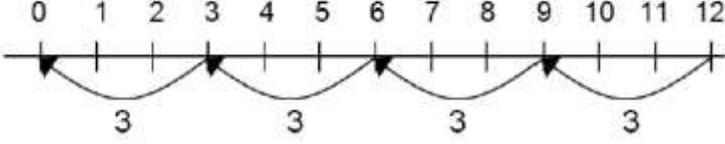
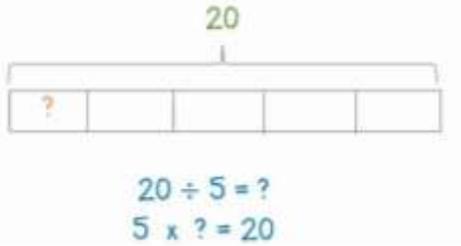
$$\begin{array}{r} 32 \\ \times 24 \\ \hline 8 \quad (4 \times 2) \\ 120 \quad (4 \times 30) \\ 40 \quad (20 \times 2) \\ \underline{600} \quad (20 \times 30) \\ 768 \end{array}$$

$$\begin{array}{r} 32 \\ \times 24 \\ \hline 8 \\ 120 \\ 40 \\ \underline{600} \\ 768 \end{array}$$

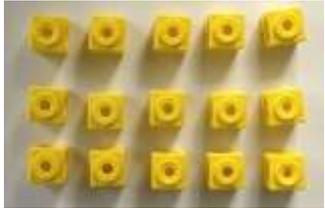
$$\begin{array}{r} 32 \\ \times 24 \\ \hline 128 \\ \underline{640} \\ 768 \end{array}$$

This moves to the more compact method.

# Division

Objective and Strategies	Concrete	Pictorial	Abstract
<p>Sharing objects into groups</p>	 <p>I have 10 cubes, can you share them equally in 2 groups?</p>	<p>Children use pictures or shapes to share quantities.</p>  <div style="border: 1px solid black; padding: 5px; width: fit-content; margin: 10px auto;"> <math>8 \div 2 = 4</math> </div>	<p>Share 9 buns between three people.</p> $9 \div 3 = 3$
<p>Division as grouping</p>	<p>Divide quantities into equal groups. Use cubes, counters, objects or place value counters to aid understanding.</p> <p><math>10 \div 5 = 2</math></p>  <p><math>35 \div 7 = 5</math></p>  <p><math>96 \div 3 = 32</math></p> 	<p>Use a number line to show jumps in groups. The number of jumps equals the number of groups. <math>12 \div 3 = 4</math></p>  <p>Think of the bar as a whole. Split it into the number of groups you are dividing by and work out how many would be within each group.</p>  <p><math>20 \div 5 = ?</math>  <math>5 \times ? = 20</math></p>	<p><math>28 \div 7 = 4</math></p> <p>Divide 28 into 7 groups. How many are in each group?</p>

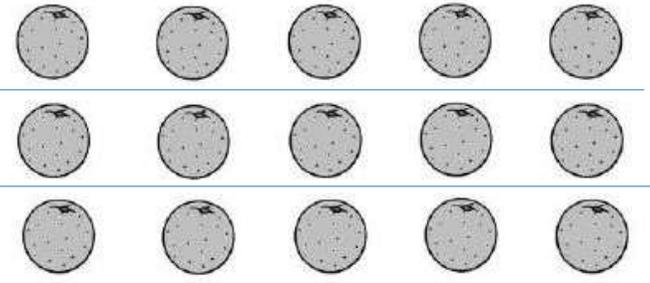
### Division within arrays



Link division to multiplication by creating an array and thinking about the number sentences that can be created.

$$15 \div 3 = 5 \quad 5 \times 3 = 15$$

$$15 \div 5 = 3 \quad 3 \times 5 = 15$$



Draw an array and use lines to split the array into groups to make multiplication and division sentences.

Find the inverse of multiplication and division sentences by creating four linking number sentences.

$$7 \times 4 = 28$$

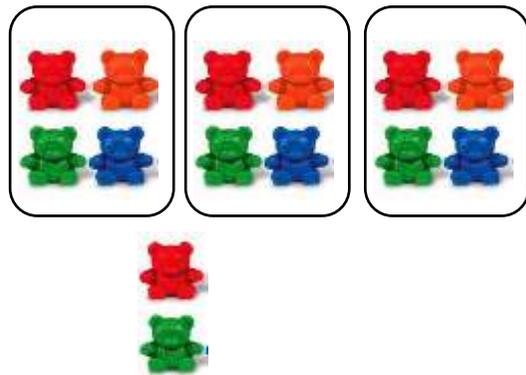
$$4 \times 7 = 28$$

$$28 \div 7 = 4$$

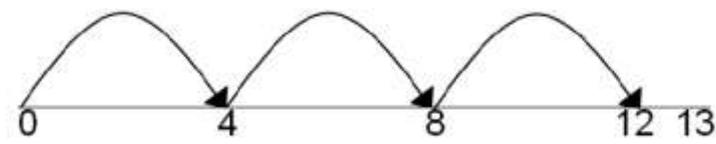
$$28 \div 4 = 7$$

### Division with a remainder

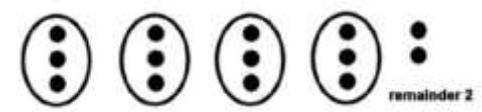
$14 \div 3 =$   
Divide objects between groups and see how much is left over



Jump forward in equal jumps on a number line then see how many more you need to jump to find a remainder.  
 $13 \div 4 = 3 \text{ r } 1$



Draw dots and group them to divide an amount and clearly show a remainder.  $14 \div 3 = 4 \text{ r } 2$ .

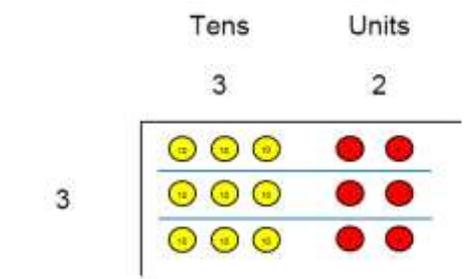


Complete written divisions and show the remainder using r (introduce related vocabulary).

$$29 \div 8 = 3 \text{ REMAINDER } 5$$

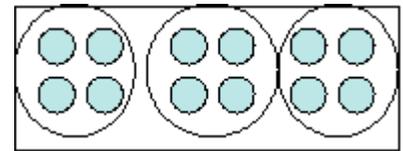
↑
↑
↑
↑
  
 dividend    divisor    quotient            remainder

### Short division



Use place value counters to divide using the bus stop method alongside  $96 \div 3 = 32$

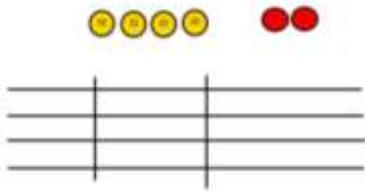
Students can continue to use drawn diagrams with dots or circles to help them divide numbers into equal groups.



Encourage them to move towards counting in multiples to divide more efficiently.  $12 \div 3 = 4$

Begin with divisions that divide equally with no remainder.

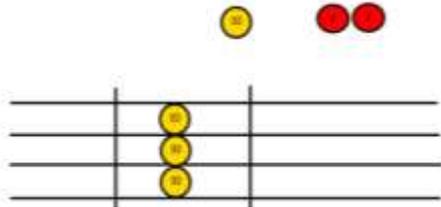
$$\begin{array}{r}
 218 \\
 4 \overline{) 872}
 \end{array}$$



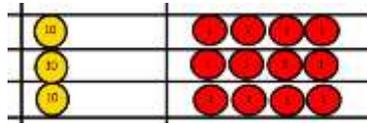
Calculations  
42 ÷ 3

$$42 \div 3 =$$

Start with the biggest place value, we are sharing 40 into three groups. We can put 1 ten in each group and we have 1 ten left over.



We exchange this ten for ten ones and then share the ones equally among the groups.



We look how much in one group so the answer is 14.

Move onto divisions with a remainder.

$$\begin{array}{r} 86 \text{ r } 2 \\ 5 \overline{) 432} \end{array}$$

Finally move into decimal places to divide the total accurately.

$$\begin{array}{r} 14.6 \\ 35 \overline{) 516.210} \end{array}$$

X	35
1	35
2	70
3	105
4	140
5	175
6	210